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A common geometrical context for the instanton, the two-dimensional O(4) sigma model and the skyrmion

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Abstract. Topological structures in gauge theories, e.g. monopoles and instantons, may be classified by either Chern classes or winding numbers. On the other hand, Wess-Zumino terms in sigma models are winding number densities and there exists a common mathematical structure for the instanton and the Wess-Zumino term in the two-dimensional O(4) sigma model. The same formalism describes the skyrmion topological Lagrangian in the four-dimensional SU(3) sigma model

1. Introduction

The past fifteen years have witnessed the rise to prominence of various topological notions in physics. Important among these are Chern classes, defined on even-dimensional spherical spaces, and winding numbers. Chern classes on S^2 are used to classify Dirac magnetic monopoles, and those on S^4 to classify instantons [1]. The $S^2 \to S^2$ winding number serves to classify the non-Abelian ('t Hooft-Polyakov) magnetic monopole [2] as well as the Belavin-Polyakov soliton [3] in the O(3) sigma model and, as was recognised by the original authors [4], the $S^3 \to S^3$ winding number describes the instanton. In more recent years Witten [5] has related the SU(3) skyrmion to the $S^5 \to \mathrm{SU}(3)$ winding number, which appears as a Wess-Zumino term and Curtright and co-workers [6] have remarked on the importance of a Wess-Zumino term in the two-dimensional sigma model, which gives rise to torsion in the internal space. This term takes the form of an $S^3 \to S^3$ winding number. In this paper we describe a general framework which relates these two topological quantities and in particular we find a natural place for torsion in the two-dimensional O(4) sigma model.

The general relation between the Chern class over S^{2n} and the winding number $S^{2n-1} \to S^{2n-1}$, or more generally $S^{2n-1} \to \mathrm{G/H}$ where G and H are Lie groups, is outlined in § 2. In § 3 this is applied to the Dirac magnetic monopole and in § 4 to the instanton. In § 5 it is shown that in a two-dimensional spacetime the Wess-Zumino term in the O(4) sigma model is an equivalent structure. In § 6 it is seen how the same analysis reveals a natural place for the skyrmion in four-dimensional spacetime. In this case, however, the Wess-Zumino term does not have the effect of endowing the internal space with torsion. The notion that there are two ways of classifying non-trivial topological structures is found in the work of Eguchi *et al* [1] and its connection with anomalies and the Skyrme model has also been clarified by Chou *et al* [7] and Zumino *et al* [8].

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2. Chern classes and winding numbers

If G is a Lie group with group element g the connection 1-form

$$\omega = g^{-1}\,\mathrm{d}g + g^{-1}Ag = g^{-1}(\mathrm{d}+A)g \tag{2.1}$$

is invariant under x-dependent group transformations $g \to hg$ so long as A transforms as

$$A \to hAh^{-1} - (\mathrm{d}h)h^{-1}. \tag{2.2}$$

A is a Lie-algebra-valued 1-form which may be written as

$$A = A^a_\mu \frac{\lambda^a}{2\mathrm{i}}\,\mathrm{d}x^\mu \tag{2.3}$$

with $\mathrm{Tr}(\lambda^a\lambda^b) = 2\delta_{ab}$. The curvature 2-form is defined by

$$\Omega = \mathrm{d}\omega + \omega^2 \tag{2.4}$$

where d is the exterior derivative operator and wedge products are implied. From (2.1) and (2.4) we have

$$\Omega = g^{-1}Fg \qquad F = \mathrm{d}A + A^2 = \tfrac{1}{2}F^a_{\mu\nu}\frac{\lambda^a}{2\mathrm{i}}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu. \tag{2.5}$$

The quantities A^a_μ and $F^a_{\mu\nu}$ are the usual Yang-Mills potential and field tensor.

Now consider the $2n$-dimensional sphere S^{2n} (see figure 1). It may be divided into upper and lower hemispheres H^{2n}_+ and H^{2n}_- which are, topologically speaking, equivalent to S^{2n} with the S pole (N pole) excised. The intersection of H^{2n}_+ and H^{2n}_- includes the 'equator' S^{2n-1}. We are concerned with Chern classes defined on S^{2n}. The total Chern form $c(\Omega)$ is defined by [1]

$$c(\Omega) = \det[1 + (\mathrm{i}/2\pi)\Omega] = 1 + \Omega_2(\Omega) + \Omega_4(\Omega) + \dots. \tag{2.6}$$

Here Ω_2 is a 2-form, Ω_4 a 4-form, etc. The first three Chern forms are

$$\begin{aligned}
&\Omega_2 = (\mathrm{i}/2\pi)\mathrm{Tr}\,F\\
&\Omega_4 = (1/8\pi^2)[\mathrm{Tr}\,F^2 - (\mathrm{Tr}\,F)^2]\\
&\Omega_6 = (\mathrm{i}/48\pi^3)[-2\,\mathrm{Tr}\,F^3 + 3(\mathrm{Tr}\,F^2)(\mathrm{Tr}\,F) - (\mathrm{Tr}\,F)^3]
\end{aligned} \tag{2.7}$$

where F has appeared rather than Ω because from (2.5) the g factor cancels out because of the trace. The Chern class c_n is simply the integral of Ω_{2n} over S^{2n}

$$c_n = \int_{S^{2n}} \Omega_{2n} \tag{2.8}$$

and it is *integral.*

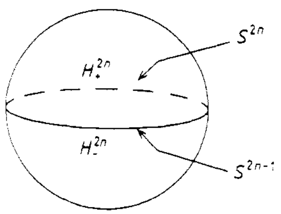

Figure 1. The $2n$-dimensional sphere S^{2n} is divided into upper and lower hemispheres H^{2n}_+ and H^{2n}_-. The region of overlap of H^{2n}_+ and H^{2n}_- includes the equator S^{2n-1}.

By virtue of the Bianchi identity the Chern forms are closed

$$d\Omega_{2n} = 0. \tag{2.9}$$

By Poincaré's lemma this means that locally Ω_{2n} can be written as the exterior derivative of a $(2n-1)$-form ω_{2n-1}, the Chern-Simons form

$$\Omega_{2n} = d\omega_{2n-1}. \tag{2.10}$$

Neither the Chern nor the Chern-Simons form is defined globally over S^{2n}; indeed, neither are F nor A. On splitting S^{2n} up into the overlapping patches H^{2n}_+ and H^{2n}_-, however, A_+ and A_-, F_+ and F_- may be defined in each patch, and each related to the other by a gauge transformation S

$$A_- = S^{-1}A_+S + S^{-1}\,dS \tag{2.11}$$

$$F_- = S^{-1}F_+S. \tag{2.12}$$

Under such gauge variations, Δ_g, the Chern-Simons form behaves as (ignoring constant coefficients)

$$\Delta_g\omega_{2n-1} = d\omega_{2n-2} + L^{2n-1} \tag{2.13}$$

where ω_{2n-2} is some $(2n-2)$-form and

$$L = S^{-1}\,dS \tag{2.14}$$

is the Cartan-Maurer form. L^{2n-1} is in fact a winding number density, so that (2.13) yields a winding number when integrated over S^{2n-1}. Because of Stokes' theorem, the integral over S^{2n-1} of the first term on the right-hand side of (2.13) vanishes since S^{2n-1} has no boundary. Equations (2.10) and (2.13) are sometimes referred to as the descent equations.

When infinitesimal transformations are considered (2.13) becomes, in the notation of Zumino *et al* [8],

$$\Delta_g\omega^0_{2n-1} = d\omega^1_{2n-2}.$$

The form ω^1_{2n-2} is related to the non-Abelian anomaly, just as Ω_{2n} is related to the chiral anomaly.

Summarising, the general relation between Chern classes and winding numbers comes from putting together the above equations (and ignoring numerical coefficients):

$$\begin{aligned} c_n &= \int_{S^{2n}} \Omega_{2n} \\ &= \int_{H^{2n}_+} \Omega^+_{2n} + \int_{H^{2n}_-} \Omega^-_{2n} \\ &= \int_{H^{2n}_+} d\omega^+_{2n-1} + \int_{H^{2n}_-} d\omega^-_{2n-1} \\ &= \int_{S^{2n-1}} \omega^+_{2n-1} - \omega^-_{2n-1} \\ &= \int_{S^{2n-1}} \Delta_g\omega^+_{2n-1} \\ &= \int_{S^{2n-1}} d\omega_{2n-2} + L^{2n-1} \\ &= \int_{S^{2n-1}} L^{2n-1}. \end{aligned} \tag{2.15}$$

The minus sign in the fourth line arises because the equator S^{2n-1} bounds H_+^{2n} and H_-^{2n} in opposite directions.

Equation (2.15) shows that there are two ways of classifying a G-bundle; as a Chern class over S^{2n}, or as a winding number over S^{2n-1}. Below we shall review, for the cases $n=1$ and $n=2$, how these descriptions apply to the magnetic monopole and the instanton. In the following sections we shall then see how they may also be applied to the Wess-Zumino terms in the non-linear sigma models, in the cases $n=2$ and $n=3$.

3. The magnetic monopole

Here $n=1$ and $G=U(1)$; we are dealing with electromagnetic potentials and field strengths defined on a sphere S^2 'surrounding the origin' (which of course does not belong to the space). From (2.5) and (2.7) the first Chern form is

$$\Omega_2 = (\mathrm{i}/2\pi)F = (\mathrm{i}/2\pi)\,\mathrm{d}A \tag{3.1}$$

hence in this particular case (where G is Abelian) A is a Chern-Simons form. On H_+^2 and H_-^2 A assumes the values

$$\begin{aligned} A_+ &= \mathrm{i}eg(1-\cos\theta)\,\mathrm{d}\phi \qquad &&\text{for } H_+^2 \\ A_- &= -\mathrm{i}eg(1+\cos\theta)\,\mathrm{d}\phi \qquad &&\text{for } H_-^2 \end{aligned} \tag{3.2}$$

corresponding to a magnetic pole of charge g. (In this section only, g stands for magnetic charge and not a group element.) A_- and A_+ are related by the gauge transformation (2.11) with

$$S = \exp(-2\mathrm{i}eg\phi) \tag{3.3}$$

then

$$A_- - A_+ = -2\mathrm{i}eg\,\mathrm{d}\phi = S^{-1}\,\mathrm{d}S. \tag{3.4}$$

In this case $F_+ = F_-$:

$$\mathrm{d}A_+ = \mathrm{d}A_- = \mathrm{i}eg\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = F. \tag{3.5}$$

The magnetic field 2-form is single-valued over S^2; this is not true in the general case. From (3.1) and (3.5) the first Chern form is

$$\Omega_2 = -(1/2\pi)eg\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

and the first Chern class is therefore

$$c_1 = -\int \Omega_2 = 2eg.$$

The requirement that c_1 is integral is the Dirac quantisation condition

$$eg = n/2. \tag{3.6}$$

This is the first way of classifying the bundle. On the other hand the Maurer-Cartan form is

$$L = -\mathrm{i}S^{-1}\,\mathrm{d}S = 2eg\,\mathrm{d}\phi$$

so the $S^1 \to S^1$ winding number is, from (2.15),

$$\int_{S^1} L = 4\pi eg = n(2\pi)$$

also yielding the Dirac condition (3.6).

4. The instanton

Here $n=2$ and $G=\mathrm{SU}(2)$; we consider SU(2) gauge fields on S^4. The matrices λ_a in (2.3) can be taken to be the Pauli matrices, so Tr $F=0$ and the second Chern class is, from (2.7) and (2.8),

$$c_2 = \frac{1}{8\pi^2} \int_{S^4} \mathrm{Tr}\, F^2. \tag{4.1}$$

This is simply minus the instanton number k

$$k = -c_2 = -\frac{1}{8\pi^2} \int_{S^4} \mathrm{Tr}\, F^2. \tag{4.2}$$

To describe the instanton by an $S^3 \to S^3$ winding number, we first find the Chern–Simons form ω_3 whose exterior derivative is Tr F^2. We have

$$\mathrm{Tr}\, F^2 = \mathrm{d}\, \mathrm{Tr}(FA - \tfrac{1}{3}A^3) \tag{4.3}$$

so that (see (2.15))

$$c_2 = \frac{1}{8\pi^2} \int_{S^3} \mathrm{Tr}(F_+A_+ - \tfrac{1}{3}A_+^3) - \mathrm{Tr}(F_-A_- - \tfrac{1}{3}A_-^3). \tag{4.4}$$

Now using (2.11) and (2.12) we have

$$\mathrm{Tr}(F_+A_+ - \tfrac{1}{3}A_+^3 - F_-A_- + \tfrac{1}{3}A_-^3) = \mathrm{Tr}\tfrac{1}{3}L^3 - \mathrm{d}(A_+L) \tag{4.5}$$

giving

$$c_2 = \frac{1}{24\pi^2} \int_{S^3} \mathrm{Tr}\, L^3 = \frac{1}{24\pi^2} \int_{S^3} \mathrm{Tr}(S^{-1}\,\mathrm{d}S)^3. \tag{4.6}$$

This expresses the Chern class as a winding number; for the BPST instanton $k=1$ so the winding number S^3 (group space) $\to S^3$ (parameter space) is (minus) one.

5. Two-dimensional O(4) sigma model

This is a scalar field theory of four scalar fields in two spacetime dimensions subject to the constraint

$$\phi_i\phi_i = \phi_0^2 + \phi_1^2 + \phi_2^2 + \phi_3^2 = 1. \tag{5.1}$$

This constraint is invariant under O(4) transformations of ϕ_i; more strictly it defines a spherical space S^3 which is a coset space O(4)/O(3). The kinetic energy Lagrangian

$$\mathscr{L}_1 = (1/2\lambda^2)\partial_\mu\phi^i\partial^\mu\phi^i \tag{5.2}$$

with $\mu = 0, 1$ and $i = 0, 1, 2, 3$ may be rewritten [9], taking into account the constraint (5.1),

$$\mathscr{L}_1 = (1/2\lambda^2) g_{ab} \partial^\mu \phi^a \partial_\mu \phi^b \tag{5.3}$$

with $a, b = 1, 2, 3$ and

$$g_{ab} = \delta_{ab} + \phi^a \phi^b / (1 - \phi^2) \tag{5.4}$$

where $\phi^2 = \phi_1^2 + \phi_2^2 + \phi_3^2$. This metric has determinant

$$g = \det g_{ab} = (1 - \phi^2)^{-1}. \tag{5.5}$$

In addition to $\mathscr{L}_1$ we may consider a topological Lagrangian in the shape of a Wess-Zumino term defined in three dimensions. This is an $S^3 \to S^3$ winding number density written as (μ, ν, $\lambda = 0, 1, 2$):

$$\mathscr{L}_{\mathrm{WZ}} = \frac{a}{24\pi^2} \int_{H^3} \sqrt{g}\, \mathrm{Tr}\, L^3 \tag{5.6}$$

where H^3 is a hemisphere of S^3 and

$$L = S^{-1}\, \mathrm{d}S \qquad S = \exp(\mathrm{i}\phi^a \sigma^a) \tag{5.7}$$

and hence

$$\mathrm{Tr}\, L^3 = 2\varepsilon^{abc}\, \mathrm{d}\phi^a\, \mathrm{d}\phi^b\, \mathrm{d}\phi^c \tag{5.8}$$

giving

$$\mathscr{L}_{\mathrm{WZ}} = \frac{a}{12\pi^2} \int \sqrt{g}\, \varepsilon^{abc}\, \mathrm{d}\phi^a\, \mathrm{d}\phi^b\, \mathrm{d}\phi^c \tag{5.9}$$

$$= \frac{a}{12\pi^2} \int \sqrt{g}\, \varepsilon^{abc} \varepsilon^{\mu\nu\lambda} \partial_\mu \phi^a \partial_\nu \phi^b \partial_\lambda \phi^c\, \mathrm{d}^3x. \tag{5.10}$$

The constant a will be found below. It will be recognised that apart from a, the right-hand side of (5.9) is an $S^3 \to S^3$ winding number. Now writing

$$\mathscr{L}_{\mathrm{WZ}} = a \int \omega_3 \tag{5.11}$$

where ω_3 is a 3-form, we search for a 2-form ω_2 with

$$\mathrm{d}\omega_2 = \omega_3 = \frac{1}{24\pi^2} \sqrt{g}\, \mathrm{Tr}\, L^3. \tag{5.12}$$

We have

$$\mathrm{d}\, \mathrm{Tr}[(\ln S) L^2] = \mathrm{Tr}\, L^3 \tag{5.13}$$

so let us assume that ω_2 can be written in the form

$$\omega_2 = (1/24\pi^2) f(\phi^2)\, \mathrm{Tr}[(\ln S) L^2] \tag{5.14}$$

$$= (1/12\pi^2) f(\phi^2) \varepsilon^{abc} \phi^a\, \mathrm{d}\phi^b\, \mathrm{d}\phi^c. \tag{5.15}$$

Equation (5.11) gives

$$2f' \phi^a\, \mathrm{d}\phi^2\, \mathrm{Tr}[(\ln S) L^2] + f\, \mathrm{Tr}\, L^3 = (1 - \phi^2)^{-1/2}\, \mathrm{Tr}\, L^3 \tag{5.16}$$

which, on using (5.8) and (5.15), yields

$$\tfrac{2}{3}f'(\phi^2)\phi^2+f(\phi^2)=(1-\phi^2)^{-1/2}. \tag{5.17}$$

The solution of this equation is

$$f(\phi^2)=\tfrac{3}{2}(\phi^2)^{-3/2}[\sin^{-1}(\phi^2)^{1/2}-(\phi^2-\phi^4)^{1/2}] \tag{5.18}$$

and the Wess-Zumino term is

$$\mathscr{L}_{\mathrm{WZ}}=\frac{a}{12\pi^2}\int_{S^2} e_{ab}\,\mathrm{d}\phi^a\,\mathrm{d}\phi^b \tag{5.19}$$

with

$$e_{ab}=\varepsilon^{abc}\phi^c f(\phi^2). \tag{5.20}$$

This result agrees with that obtained by Curtright and co-workers [6] who call e_{ab} a 'torsion potential' for reasons to be seen below.

The Wess-Zumino action is multivalued, as was first pointed out in this context by Witten [5]. In continuing the action in (5.19) from two-dimensional spacetime (S^2) to three dimensions, we may continue to either H^3_+ or H^3_- and these cases will differ by a minus sign, since S^2 bounds them in different directions. Quantum theory involves $\exp(\mathrm{i}S)$ so the continuation is unambiguous if

$$\exp\left(\mathrm{i}\int_{H_+}\mathscr{L}_{\mathrm{WZ}}\right)=\exp\left(-\mathrm{i}\int_{H_-}\mathscr{L}_{\mathrm{WZ}}\right)$$

or

$$\int_{S^3}\mathscr{L}_{\mathrm{WZ}}=2\pi N \tag{5.21}$$

where $S^3=H_+\cup H_-$ and N is an integer. If we normalise $\mathscr{L}_{\mathrm{WZ}}$ so that

$$\int_{S^3}\omega_3=1 \tag{5.22}$$

then (5.21) gives

$$a=2\pi N. \tag{5.23}$$

The total action is then

$$\int\mathscr{L}=\int(\mathscr{L}_1+\mathscr{L}_{\mathrm{WZ}})=\frac{1}{2\lambda^2}\int[g_{ab}\varepsilon^{\mu\nu}\phi^a_{,\mu}\phi^b_{,\nu}+\tfrac{2}{3}\eta e_{ab}\varepsilon^{\mu\nu}\phi^a_{,\mu}\phi^b_{,\nu}]\,\mathrm{d}^2x \tag{5.24}$$

with

$$\eta=N\lambda^2/2\pi. \tag{5.25}$$

It is straightforward to show that the equation of motion resulting from (5.24) is

$$(\delta^{ab}\partial_\mu+\Gamma^a_{bc}\phi^c_{,\mu}+\varepsilon_{\mu\nu}S^a_{bc}\phi^c_{,\nu})\phi^b_{,\mu}=0 \tag{5.26}$$

where

$$S^a_{bc}=g^{ad}S_{dbc}\qquad S_{dbc}=\tfrac{1}{3}\eta(e_{db,c}-e_{cb,d}-e_{dc,b}). \tag{5.27}$$

Equation (5.26) is a geodesic equation with $\Gamma^a_{bc}+S^a_{bc}$ playing the role of a connection coefficient. The antisymmetry of S^a_{bc} indicates that it behaves as a *torsion tensor.* From (5.17) and (5.20) we have

$$S^{abc}=\eta g^{-1/2}\varepsilon^{abc} \tag{5.28}$$

which, since ε^{abc} is a permutation symbol, has the transformation properties of a contravariant tensor.

We conclude this section by giving an axiomatic treatment of torsion in group space in the non-linear sigma model, and by reproducing the results of Curtright *et al* on parallelisable manifolds. In our group space the basis 1-form is $\mathrm{d}\phi^a$

$$e^a=\mathrm{d}\phi^a. \tag{5.29}$$

There is a connection 1-form ω^a_b and a torsion 2-form τ^a defined by

$$\tau^a=\mathrm{d}e^a+\omega^a_b e^b. \tag{5.30}$$

In the absence of torsion $\tau^a=0$ and ω^a_b is determined in terms of e^a. In any case the curvature 2-form is

$$R^a_b=\mathrm{d}\omega^a_b+\omega^a_c\omega^c_b. \tag{5.31}$$

The Cartan-Maurer form (5.7) becomes

$$L=e^aT^a \tag{5.32}$$

and since it obeys $\mathrm{d}L=-L^2$ we have

$$\mathrm{d}e^c=-\tfrac{1}{2}\varepsilon^{abc}e^ae^b. \tag{5.33}$$

In the absence of torsion (5.30) then gives

$$\omega^{ab}=-\tfrac{1}{2}\varepsilon^{abc}e_c \qquad \tau^\alpha=0. \tag{5.34}$$

As a simple *ansatz*, let us suppose, with Wu and Zee [10], that in the *presence* of torsion this simply becomes modified to

$$\omega^{ab}=-\tfrac{1}{2}K\varepsilon^{abc}e_c. \tag{5.35}$$

We now want to find the values of K which give a *parallelised* manifold, i.e. one with $R_{ab}=0$. Substituting (5.35) into (5.31) and using (5.33) gives

$$R^{ab}=-\tfrac{1}{2}K\varepsilon^{abc}-\tfrac{1}{2}\varepsilon^{cde}e^de^e+\tfrac{1}{4}K^2\varepsilon^{acd}\varepsilon^{cbe}e^de^e. \tag{5.36}$$

Applying the Jacobi identity to the second term in R_{ab} and using the fact that $R^{ab}=-R^{ba}$ gives, for this term,

$$R_2^{ab}=(K^2/8)\varepsilon^{abc}\varepsilon^{cde}e^de^e$$

and hence

$$R^{ab}=\tfrac{1}{4}K(1-\tfrac{1}{2}K)\varepsilon^{abc}\varepsilon^{cde}e^de^e. \tag{5.37}$$

We see that $R^{ab}=0$ if $K=0$ or $K=2$, giving respectively

$$\omega^{ab}=0 \qquad \tau^a=-\tfrac{1}{2}\varepsilon^{abc}e^be^c \qquad \text{for } K=0 \tag{5.38}$$

$$\omega^{ab}=-\varepsilon^{abc}e_c \qquad \tau^a=\tfrac{1}{2}\varepsilon^{abc}e^be^c \qquad \text{for } K=2. \tag{5.39}$$

These two cases correspond to a torsion tensor

$$S^{abc}=\pm g^{-1/2}\varepsilon^{abc}. \tag{5.40}$$

The sign ambiguity corresponds to the values of the torsion on $H^3_\pm$, since ω_3 is not defined globally on S^3. Comparing (5.40) and (5.28) we see that we get a parallelised manifold if $\eta = 1$, i.e. if

$$\lambda^2 = 2\pi/N \tag{5.41}$$

as found in [6]. This may be regarded as another type of quantisation, resulting from the multivalued Wess-Zumino action.

It may be worth remarking that the reason the torsion appears directly in the action in this theory is that the theory is in two spacetime dimensions. Torsion is a 2-form and a 2-form in field space (as in (5.19)) can be transposed into a 2-form in spacetime (as in (5.27)) when spacetime has two dimensions. The torsion contribution to the action in general relativity is of course quite different, since the action there must be a 4-form. Similarly, in the four-dimensional SU(3) skyrmion model to be discussed below, the effect of torsion in the group manifold does not have such a direct connection with the Wess-Zumino term as it has in the two-dimensional O(4) sigma model.

6. S^6 and the SU(3) skyrmion

The Skyrme model [11], in which baryons are represented as solitons in a non-linear sigma model, has undergone a revival in recent years through the work of Witten [5] who showed that the Wess-Zumino term in four dimensions ensured that the solitons (skyrmions) are fermions. The Wess-Zumino term is an $S^5 \to S^5$, or more precisely $S^5 \to \mathrm{SU}(3)$ (the group space of SU(3) being locally isomorphic to $S^5 \times S^3$), winding number. It therefore fits naturally into our scheme; that is, we consider an SU(3) gauge field on S^6, which may be characterised by a third Chern class or an $S^5 \to S^5$ winding number, as in (2.15). This observation has already been made by Chou and coworkers [7]. It is included here briefly to demonstrate its essential similarity with the two-dimensional O(4) sigma model and thereby to give a more explicit form of the Wess-Zumino term.

In the notation of § 2, $n = 3$ and A and F are SU(3) matrix fields. The third Chern class is from (2.7)

$$c_3 = \frac{-\mathrm{i}}{24\pi^3} \int_{S^6} \mathrm{Tr}\, F^3 \tag{6.1}$$

since $\mathrm{Tr}\, F = 0$. As usual, S^6 is decomposed into the hemispheres H^6_+ and H^6_- on each of which F and A are defined, giving

$$c_3 = \frac{-\mathrm{i}}{24\pi^3} \left(\int_{H^6_+} \mathrm{Tr}\, F^3_+ + \int_{H^6_-} \mathrm{Tr}\, F^3_- \right). \tag{6.2}$$

The 5-form of which $\mathrm{Tr}\, F^3$ is the exterior derivative (locally) is given by

$$\mathrm{Tr}\, F^3 = \mathrm{d}\, \mathrm{Tr}(F^2 A - \tfrac{1}{2} F A^3 + \tfrac{1}{10} A^5) \tag{6.3}$$

so

$$c_3 = \frac{-\mathrm{i}}{24\pi^3} \int_{S^5} \mathrm{Tr}(F^2_+ A_+ - \tfrac{1}{2} F_+ A^3_+ + \tfrac{1}{10} A^5_+ - F^2_- A_- + \tfrac{1}{2} F_- A^3_- - \tfrac{1}{10} A^5_-) \tag{6.4}$$

when A_+, A_- and F_+, F_- are related by (2.11) and (2.12). We now expect that the 5-form in the integrand of (6.4) is the exterior derivative of a 4-form together with a winding number $\mathrm{Tr}\, L^5$. After some algebra we obtain

$$c_3 = \frac{-\mathrm{i}}{24\pi^3} \int_{S^5} (\mathrm{d}\omega_4 + \tfrac{1}{10}\,\mathrm{Tr}\, L^5) \tag{6.5}$$

where

$$\omega_4 = \mathrm{Tr}(A_+\, \mathrm{d}\, A_+ L + \tfrac{1}{2}\, A_+^3 L + \tfrac{1}{2}\, A_+^2 L^2 + \tfrac{1}{4}\, A_+ L A_+ L - \tfrac{1}{2}\, A_+ L^3). \tag{6.6}$$

The ω_4 contribution vanishes by Stokes' theorem and finally

$$c_3 = \frac{-\mathrm{i}}{240\pi^3} \int_{S^5} \mathrm{Tr}\, L^5 \tag{6.7}$$

where

$$L = U^{-1}\, \mathrm{d}U \tag{6.8}$$

and

$$U = \exp \frac{\mathrm{i}}{F_\pi}\, \phi^a \lambda^a \tag{6.9}$$

is an element of SU(3). In the SU(3) sigma model the kinetic energy Lagrangian

$$\mathscr{L}_1 = \tfrac{1}{4} F_\pi^2\, \mathrm{Tr}(\partial^\mu U \partial_\mu U^+) = -\tfrac{1}{4} F_\pi^2\, \mathrm{Tr}\, L_\mu L^\mu \tag{6.10}$$

describes low-energy current algebra with $F_\pi \sim 190$ MeV, but it does not describe anomalies and it possesses too much symmetry [5]. The addition of a Wess-Zumino (winding number) term solves these problems. This term is of the form of (6.7) above and may be written

$$\mathscr{L}_{\mathrm{WZ}} = \frac{a}{120\pi^3} \int \sqrt{g}\, \varepsilon^{abcde}\, \mathrm{d}\phi^a \ldots \mathrm{d}\phi^e. \tag{6.11}$$

We have, ignoring F_π,

$$\mathrm{Tr}\, L^5 = \mathrm{i}^5\, \mathrm{Tr}\, \lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5 \varepsilon^{abcde}\, \mathrm{d}\phi^a \ldots \mathrm{d}\phi^e = -\mathrm{i}\varepsilon^{abcde}\, \mathrm{d}\phi^a \ldots \mathrm{d}\phi^e$$

giving

$$\mathscr{L}_{\mathrm{WZ}} = \frac{\mathrm{i}a}{120\pi^2} \int_{H^5} \sqrt{g}\, \mathrm{Tr}\, L^5 \tag{6.12}$$

where H^5 is a hemisphere of S^5. As in the two/three-dimensional case, this action is multivalued unless $a = 2\pi N$, so we have

$$\mathscr{L}_{\mathrm{WZ}} = \frac{\mathrm{i}N}{60\pi^2} \int_{H^5} \sqrt{g}\, \mathrm{Tr}\, L^5 = \frac{\mathrm{i}N}{60\pi^2} \int_{H^5} \omega_5. \tag{6.13}$$

To express this as an integral over spacetime we want a 4-form ω whose 4 exterior derivative is ω_5

$$\mathrm{d}\omega_4 = \omega_5 = \sqrt{g}\, \mathrm{Tr}\, L^5.$$

Putting

$$\omega_4 = h(\phi^2)\, \mathrm{Tr}[(\ln U) L^4]$$

we see that $h(\phi)$ satisfies the equation

$$\tfrac{2}{5}h'(\phi^2)\phi^2+h(\phi^2)=(1-\phi^2)^{-1/2}. \tag{6.14}$$

This is of the same type as (5.17) and its solution is

$$h(\phi^2)=\tfrac{5}{2}(\phi^2)^{-5/2}[\sin^{-1}(\phi^2)^{1/2}-(\phi^2-\phi^4)^{1/2}]. \tag{6.15}$$

Thus we find

$$\mathscr{L}_{\mathrm{WZ}}=\frac{\mathrm{i}N}{60\pi^2}\int_{S^4} h(\phi^2)\varepsilon^{abcde}\phi^a\,\mathrm{d}\phi^b\,\mathrm{d}\phi^c\,\mathrm{d}\phi^d\,\mathrm{d}\phi^e \tag{6.16}$$

which is of the form originally suggested by Wess and Zumino [12].

7. Conclusion

We have shown how the topological notions of Chern class and winding numbers find an application in a number of physical situations. In gauge theories in two and four dimensions they describe magnetic monopoles and instantons. On the other hand the formalism of gauge theories in four and six dimensions is seen to yield the Wess–Zumino term in non-linear sigma models in two and four dimensions, giving in the former case torsion in the internal manifold and in the latter case the SU(3) topological skyrmion term. These results are summarised in table 1.

Table 1. Topological structures in gauge theories and σ models.

	Two dimensions	Four dimensions	Six dimensions
U(1)	Monopole[a]		
SU(2)		instanton[b] $\mathscr{L}_{\mathrm{WZ}}$ in O(4) σ model in two dimensions[c]	
SU(3)			$\mathscr{L}_{\mathrm{WZ}}$ in SU(3) σ model in four dimensions[d]

[a] Because $\pi_1(\mathrm{U}(1))=Z$.
[b] Because $\pi_3\,(\mathrm{SU}(2))=Z$.
[c] Because coset space $\mathrm{O}(4)/\mathrm{O}(3)=S^3$.
[d] Because coset space $\mathrm{SU}(3)/\mathrm{SU}(2)=S^5$.

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